

# Adler Function, DIS sum rules and Crewther Relations \*

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The current status of the Adler function and two closely related Deep Inelastic Scattering (DIS) sum rules, namely, the Bjorken sum rule for polarized DIS and the Gross-Llewellyn Smith sum rule are briefly reviewed. A new result is presented: an analytical calculation of the coefficient function of the latter sum rule in a generic gauge theory in order  $\mathcal{O}(\alpha_s^4)$ . It is demonstrated that the corresponding Crewther relation allows to fix two of three colour structures in the  $\mathcal{O}(\alpha_s^4)$  contribution to the singlet part of the Adler function.

## 1. Introduction

Exactly ten years ago at “Loops and Legs 2000” [1] one of the present authors discussed the perspectives of computing the famous ratio  $R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$  or, equivalently, the Adler function of the correlator of the EM vector currents at order  $\alpha_s^4$  in massless QCD (for a general review see [2]). The main conclusion was that “... a better understanding of all kinds of relations connecting various p-integrals could eventually result to the reduction of an arbitrary 5-loop p-integral to a combination of some limited number (a few dozens?) of master p-integrals<sup>3</sup>. Once it is done it should be not very difficult to evaluate the latter analytically or numerically ...” Now, ten years later, we can rightfully state that the program has been successfully worked out and almost (see below) completed. [3,4,5,6,7,8].

Below we summarize the current status of the calculations of the Adler function and of two other, through generalized Crewther relations

[9,10] closely related, physical observables: the Bjorken and the Gross-Llewellyn Smith DIS sum rules [11,12] at order  $\alpha_s^4$  in massless QCD.

## 2. Adler Function

It is convenient to start with the polarization operator of the flavor singlet vector current:

$$3Q^2\Pi(Q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T j_\mu(x) j^\mu(0) | 0 \rangle, \quad (1)$$

with  $j_\mu = \sum_i \bar{\psi}_i \gamma_\mu \psi_i$  and  $Q^2 = -q^2$ . The corresponding Adler function

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2) \quad (2)$$

is naturally decomposed into a sum of the non-singlet (NS) and singlet (SI) components (see Fig. 1):

$$D(Q^2) = n_f D^{NS}(Q^2) + n_f^2 D^{SI}(Q^2). \quad (3)$$

Here  $n_f$  stands for the total number of quark flavours; all quarks are considered as massless.

Note that the Adler function  $D^{EM}$  corresponding to the electromagnetic vector current  $j_\mu^{EM} = \sum_i Q_i \bar{\psi}_i \gamma_\mu \psi_i$  ( $Q_i$  stands for the electric charge of the quark field  $\psi_i$ ) is given by the expression:

$$D^{EM} = \left( \sum_i Q_i^2 \right) D^{NS} + \left( \sum_i Q_i \right)^2 D^{SI}. \quad (4)$$

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<sup>3</sup>By “p-integrals” we understand massless propagator-like (that is dependent on only one external momentum) Feynman integrals.

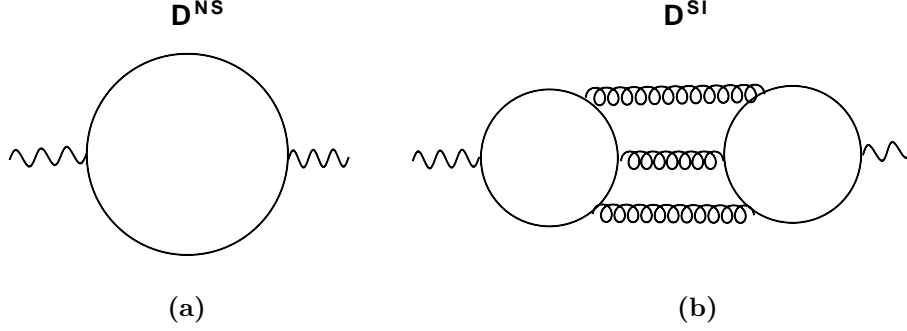


Figure 1. (a),(b): lowest order nonsinglet and singlet diagrams contributing to the Adler function.

The perturbative expansions of nonsinglet and singlet parts read ( $a_s \equiv \frac{\alpha_s}{\pi}$ ):

$$D^{NS}(Q^2) = d_R \left( 1 + \sum_{i=1}^{\infty} d_i^{NS} a_s^i(Q^2) \right), \quad (5)$$

$$D^{SI}(Q^2) = d_R \left( \sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right), \quad (6)$$

where for future convenience the parameter  $d_R$  (the dimension of the quark color representation,  $d_R = 3$  in QCD) is factorized in *both* nonsinglet and singlet components.

At order  $\alpha_s^3$  both components of the Adler function are known since long [13,14,15] for the case of a general colour gauge group. The corresponding  $\alpha_s^4$  calculation has been recently finished [5,6] for the nonsinglet function  $D^{NS}$  only.

The singlet component has the following structure at orders  $\alpha_s^3$  and  $\alpha_s^4$ :

$$d_3^{SI} = \frac{d^{abc}d^{abc}}{d_R} \left( \frac{11}{192} - \frac{1}{8}\zeta_3 \right), \quad (7)$$

$$d_4^{SI} = \frac{d^{abc}d^{abc}}{d_R} (C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI}). \quad (8)$$

Here  $C_F$  and  $C_A$  are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra,  $d^{abc} = 2 \text{Tr}(\{\frac{\lambda^a}{2}\frac{\lambda^b}{2}\}\frac{\lambda^c}{2})$ ,  $T$  is the trace normalization of the fundamental

representation. For QCD (colour gauge group  $SU(3)$ ):

$$C_F = 4/3, C_A = 3, T = 1/2, d^{abc}d^{abc} = \frac{40}{3}.$$

From general considerations we expect that the missing  $\alpha_s^4$  contribution to the singlet component of the Adler function should be numerically inessential (see the next section for an explicit argument in favour of this assumption). The corresponding direct calculation is under way and should be finished in the near future.

### 3. DIS sum rules

The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function  $C^{Bjp}$ :

$$\begin{aligned} \Gamma_1^{p-n}(Q^2) &= \int_0^1 [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)] dx \\ &= \frac{g_A}{6} C^{Bjp}(a_s) + \sum_{i=2}^{\infty} \frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, \end{aligned} \quad (9)$$

where  $g_1^{ep}$  and  $g_1^{en}$  are the spin-dependent proton and neutron structure functions,  $g_A$  is the nucleon axial charge as measured in neutron  $\beta$ -decay. The coefficient function  $C^{Bjp}(a_s) = 1 + \mathcal{O}(a_s)$  is proportional to the flavour-nonsinglet axial vector current  $\bar{\psi}\gamma^\mu\gamma_5 t^a\psi$  in the corresponding short distance Wilson expansion. The sum in the second line of (9) describes the nonperturbative power

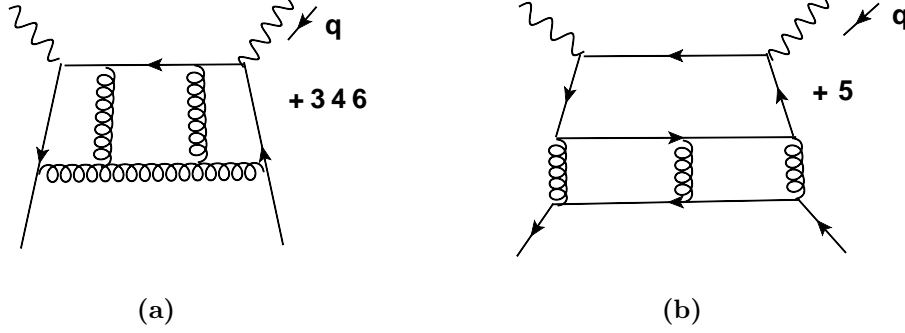


Figure 2. (a),(b):  $\mathcal{O}(\alpha_s^3)$  nonsinglet and singlet diagrams contributing to the Gross-Llewellyn Smith sum rule; note that the coefficient function  $C^{Bjp}$  is contributed by only nonsinglet diagrams.

corrections (higher twist) which are inaccessible for pQCD.

Another, closely related sum rule, the Gross-Llewellyn Smith one, reads (we do not write explicitly the higher twist corrections below)

$$\frac{1}{2} \int_0^1 F_3^{\nu p + \bar{\nu} p}(x, Q^2) dx = 3 C^{GLS}(a_s), \quad (10)$$

where  $F_3^{\nu p + \bar{\nu} p}(x, Q^2)$  is the isospin singlet structure function. The function  $C^{GLS}(a_s)$  comes from operator-product expansion of the axial and vector nonsinglet currents

$$i \int T A_\mu^a(x) V_\nu^b(0) e^{iqx} dx|_{q^2 \rightarrow \infty} \approx C_{\mu\nu\alpha}^{V,ab} V_\alpha(0) + \dots \quad (11)$$

where

$$C_{\mu\nu\alpha}^{V,ab} \sim i \delta^{ab} \epsilon_{\mu\nu\alpha\beta} \frac{q^\beta}{Q^2} C^{GLS}(a_s)$$

and  $V_\alpha = \bar{\psi} \gamma_\alpha \psi$  is a flavour singlet quark current. At last  $A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 t^a \psi$ ,  $V_\nu^b = \bar{\psi} \gamma_\nu t^b \psi$  are axial vector and vector nonsinglet quark currents, with the  $t^a$ ,  $t^b$  being the generators of the flavour group  $SU(n_f)$ .

All diagrams contributing to  $C^{GLS}(a_s)$  can be separated in two groups: nonsinglet and singlet ones (see Fig. 2):

$$C^{GLS} = C^{NS} + C^{SI}, \quad (12)$$

$$C^{NS}(Q^2) = 1 + \sum_{i=1}^{\infty} c_i^{NS} a_s^i(Q^2), \quad (13)$$

$$C^{SI}(Q^2) = \sum_{i=3}^{\infty} c_i^{SI} a_s^i(Q^2). \quad (14)$$

The results for both functions  $C^{Bjp}$  and  $C^{GLS}$  at order  $\alpha_s^3$  are known since early 90-ties [17]. Note that there is a remarkable connection (following from the chiral invariance [17])

$$C^{Bjp} \equiv C^{NS} \quad (15)$$

valid in all orders of the perturbation theory.

The  $\mathcal{O}(\alpha_s^4)$  contribution to  $C^{Bjp}$  has been recently computed and published in [6]. The calculation of the  $\mathcal{O}(\alpha_s^4)$  contribution to  $C^{SI}$  has been just finished. The results for both functions are given below.

We start from their numerical form.

$$\begin{aligned} C^{NS} = & 1 - a_s + (-4.583 + 0.3333 n_f) a_s^2 \\ & + a_s^3 (-41.44 + 7.607 n_f - 0.1775 n_f^2) a_s^3 \\ & + (-479.4 + 123.4 n_f - 7.697 n_f^2 + 0.104 n_f^3) a_s^4, \end{aligned} \quad (16)$$

$$C^{SI} = 0.4132 n_f a_s^3 + a_s^4 n_f (5.802 - 0.2332 n_f). \quad (17)$$

For a typical value of  $n_f = 3$  the above relations read:

$$\begin{aligned} C^{NS}(n_f = 3) = & 1 - 1. a_s - 3.583 a_s^2 \\ & - 20.22 a_s^3 - 175.7 a_s^4, \end{aligned} \quad (18)$$

$$C^{SI} = 1.2396 a_s^3 + 15.3072 a_s^4. \quad (19)$$

As expected the singlet contributions are less than the nonsinglet ones by at least one order in

magnitude for each from two available orders in  $\alpha_s$ .

The result for  $C^{NS}$  valid for a generic gauge group is given in [6]. For the singlet coefficient function the generalization of eq. (17) to a generic gauge group read

$$c_3^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} \left( c_{3,1}^{SI} \equiv -\frac{11}{192} + \frac{1}{8} \zeta_3 \right), \quad (20)$$

$$c_4^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} (C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T n_f c_{4,3}^{SI}), \quad (21)$$

$$c_{4,1}^{SI} = \frac{37}{128} + \frac{1}{16} \zeta_3 - \frac{5}{8} \zeta_5, \quad (22)$$

$$c_{4,2}^{SI} = -\frac{481}{1152} + \frac{971}{1152} \zeta_3 - \frac{295}{576} \zeta_5 + \frac{11}{32} \zeta_3^2, \quad (23)$$

$$c_{4,3}^{SI} = \frac{119}{1152} - \frac{67}{288} \zeta_3 + \frac{35}{144} \zeta_5 - \frac{1}{8} \zeta_3^2. \quad (24)$$

#### 4. Crewther relations

There exist two (generalized) Crewther relations which connect the nonsinglet and the full Adler functions to the coefficient functions  $C^{Bjp}$  and  $C^{GLS}$  respectively [10]. The relations state that

$$D^{NS}(a_s) C^{Bjp}(a_s) = d_R \left[ 1 + \frac{\beta(a_s)}{a_s} K^{NS} \right], \quad (25)$$

$$K^{NS} \equiv K^{NS}(a_s) = a_s K_1^{NS} + a_s^2 K_2^{NS} + a_s^3 K_3^{NS} + \dots$$

and

$$\frac{D(a_s) C^{GLS}(a_s)}{d_R n_f} = 1 + \frac{\beta(a_s)}{a_s} K(a_s), \quad (26)$$

$$K(a_s) = a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \dots$$

Here  $\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i \geq 0} \beta_i a_s^{i+2}$  is the QCD  $\beta$ -function with its first term  $\beta_0 = \frac{11}{12} C_A - \frac{T}{3} n_f$ . The term proportional to the  $\beta$ -function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order  $\alpha_s^2$ . The relations (25) and (26) were suggested in [10] on the basis of

[9,18] and the  $\mathcal{O}(\alpha_s^3)$  calculations of the functions  $C^{Bjp}$ ,  $C^{GLS}$  and  $D$  carried out in [13,14,17]. Formal proofs were considered in [19,20].

Relation (25) has been investigated in work [6]. Here it was demonstrated that at orders  $\alpha_s^2, \alpha_s^3$  and  $\alpha_s^4$  eq. (25) produces as many as 2, 3 and, finally, 6 constraints on the combinations  $d_2^{NS} + c_2^{NS}, d_3^{NS} + c_3^{NS}$  and  $d_4^{NS} + c_4^{NS}$  respectively (for a very detailed discussion at orders  $\alpha_s^2$  and  $\alpha_s^3$  see also [10]).

The fulfillment of these constraints has provided us with a powerful test of the correctness of the calculations of  $D^{NS}(a_s)$  and  $C^{Bjp}(a_s)$ . It also fixes unambiguously the (nonsinglet) Crewther parameters  $K_1^{NS}, K_2^{NS}$  and  $K_3^{NS}$  (for explicit expressions see [6]).

Let us consider now eq. (26) (assuming that (25) is fulfilled). Combining eqs. (3,12,15) and (25) leads to the following relations between coefficients  $K_i^{NS}$  and  $K_i$ :

$$K_1 = K_1^{NS}, \quad K_2 = K_2^{NS}, \quad (27)$$

$$K_3 = K_3^{NS} + K_3^{SI}, \quad (28)$$

$$K_3^{SI} = k_{3,1}^{SI} n_f \frac{d^{abc} d^{abc}}{d_R}. \quad (29)$$

Thus, we conclude that eq. (26) puts  $3 - 1 = 2$  constraints between two triplets of (purely numerical) parameters  $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$  and  $\{c_{4,1}^{SI}, c_{4,2}^{SI}, c_{4,3}^{SI}\}$  appearing in eqs. (8) and (21) and completely describing the order  $\alpha_s^4$  singlet contributions to the Adler function and the Gross-Llewellyn Smith sum rule respectively.

The solution of the constraints and eqs. (22-24) produces the following result for  $d_4^{SI}$ :

$$d_{4,1}^{SI} = -\frac{3}{2} c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad (30)$$

$$d_{4,2}^{SI} = -c_{4,2}^{SI} - \frac{11}{12} k_{3,1}^{SI}, \quad (31)$$

$$d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3} k_{3,1}^{SI}. \quad (32)$$

#### 5. Conclusion

We have analytically computed the  $\mathcal{O}(\alpha_s^4)$  contribution to the Gross-Llewellyn Smith sum rule.

The result taken together with the corresponding (generalized) Crewther relation leads to a prediction for the (still unknown)  $\mathcal{O}(\alpha_s^4)$  term in the singlet component of the Adler function. The prediction depends on only one unknown numerical parameter — the Crewther coefficient  $k_{3,1}^{SI}$ .

The direct calculation of the three coefficients  $\{d_{4,1}^{SI}, d_{4,2}^{SI}, d_{4,3}^{SI}\}$  parameterizing the  $\mathcal{O}(\alpha_s^4)$  contribution to the singlet component of the Adler function will be finished soon. Then will get another strong check of the complicated machinery employed in performing the calculations.

The calculation of the coefficient function  $C^{GLS}$  has been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel MPI-based [21] as well as thread-based [22] versions of FORM [23]. For the evaluation of color factors we have used the FORM program *COLOR* [24]. The diagrams have been generated with QGRAF [25]. The figures have been drawn with the the help of Axodraw [26] and JaxoDraw [27].

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